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Abstract

This paper introduces a class of games called the positive semidefinite games, for which we show the absence of mixed and nonstrict ESS's. As a result, a strategy is an ESS if and only if it is strict Nash. One famous example in this class of games is Rock–Paper–Scissors. For a smaller class of games called the positive definite games, we prove a similar result for NSS's. This result opens the door to a corollary: for doubly symmetric games, the existence of an ESS is assured. This is an interesting result because of the stronger dynamic stability properties of ESS's as compared to NSS's. The coordination games played on the identity matrix are an example of games in this latter class.

1 Introduction

Given any payoff matrix, there are two evident questions about evolutionary stability criteria which may be asked:

1. When do ESS's/NSS's exist?
2. Are the ESS's/NSS's in a game mixed or pure?

By answering the second question for special classes of games, we will pave the way to answer the first question. This then tells us more about the usefulness of evolutionary stability criteria. The approach pursued here is different from existing literature: instead of giving conditions for the existence of pure ESS's, we present a class of games for which we show the nonexistence of mixed ESS's. This entails that the only candidates for ESS's are the pure strategies. For a smaller class of games, we strengthen this result to the absence of mixed NSS's. Furthermore, we demonstrate that when checking whether a pure strategy under consideration is indeed an ESS, we can restrict ourselves to pure strategies which is not true in general. This is convenient, because we can bring back the search space from the simplex (uncountably many strategies) to its vertices (finitely many strategies). We show that the results presented here match with

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the literature and that we can reinforce one consequence mentioned in Weibull [1995, p. 56, Proposition 2.14].

This paper is organized as follows. Section 2 presents the definitions and notations used throughout the paper, in particular, it introduces the positive (semi)definite games to the reader. Section 3 reviews the existing literature on the subject. Section 4 contains the results about absence/presence of ESS's and NSS's and some corollaries. Section 5 concludes.

2 Definitions and Notation

We will consider symmetric games and assume that every player disposes of $n \geq 2$ pure strategies, although all claims remain valid for 1×1 "games" because any claim about pure strategies, ESS's etc. is vacuously true in such cases (as is easily verified). A two-player symmetric game can be represented by an $n \times n$ matrix $A = (a_{ij})_{i,j=1,\dots,n}$ with the interpretation $X = Y = \{1, \dots, n\}$ and $\forall (i,j) \in X \times Y$ $u_1(i,j) = u_2(j,i) = a_{ij}$. In what follows we consider only this class of games. Any game in this class will be represented by the matrix A, and will be referred to by, simply, a game. The set of mixed strategies will be denoted by the simplex $\Delta_n = \{x \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i = 1\}$, whose vertices (the pure strategies) are denoted by $e^1, \dots, e^n$. When there is no risk of confusion, we will drop the subscript and write simply Δ instead of Δ_n. It is clear that the mixed extension of a symmetric game is itself symmetric. We shall only consider Nash equilibria that are symmetric.

Let S be a nonempty subset of $\{1, 2, \dots, n\}$. We will use the notation A^S as the restriction of the game A on the selected strategies in S. Analogously, x^S denotes the restriction of strategy x on the elements in S. Notice that by doing so, the components of x^S may or may not add up to 1 in general and so x^S may or may not be interpreted as a strategy under most general circumstances.

We use $\mathbf{1}_S$ as the indicator function of the set S and use $z_i^+ = \max(0, z_i)$ and $z_i^- = \max(0, -z_i)$. For any $x, y \in \Delta$, the payoff (of player 1) is $u(x,y) = x^T A y$. We will use the following definitions of an Evolutionarily/Neutrally Stable Strategy (see e.g. Weibull [1995]).

Definition 1 (Evolutionarily/Neutrally Stable Strategy)**.** *$x \in \Delta$ is an evolutionarily stable strategy/ESS (respectively neutrally stable strategy/NSS) if it fulfills conditions 1 and 2 (respectively 1 and 2')*

1. $u(y,x) \leq u(x,x) \qquad \forall y$
2. $u(y,x) = u(x,x) \Rightarrow u(y,y) < u(x,y) \qquad \forall y \neq x$

2'. $u(y,x) = u(x,x) \Rightarrow u(y,y) \leq u(x,y) \qquad \forall y$

Following Weibull [1995], we denote the (possibly empty) set of ESS's (NSS's) by Δ^{ESS} (Δ^{ESS}).

Our attention will go out to the special class of games, which we will call "positive semidefinite" and "positive definite" games (this terminology comes from "negative/positive definite games" in Hofbauer [2011]). We need the following set (see again Hofbauer [2011]):

$$\mathbb{R}^n_0 := \left\{ z \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n z_i = 0 \right\}$$

A game A is said to be positive semidefinite if $x^T Ax \geq 0$ for all $x \in \mathbb{R}_0^n$. It is said to be positive definite if it satisfies the stronger condition $x^T Ax > 0$ for all $x \in \mathbb{R}_0^n \backslash \{0\}$.

3 Literature Review

Early attention for the relation between payoff matrices and evolutionarily stable strategies appeared in an article by Haigh [1975] and a subsequent paper by Bishop and Cannings [1976]. Abakuks [1980] provided corrections to the two articles. The goal of these articles was to describe an algorithm for finding ESS's. A necessary and sufficient condition for $x \in \Delta$ to fulfill the first condition of an ESS is that the following holds for every i that is assigned positive probability:

$$(Ax)_i = \max_{1 \leq j \leq n} (Ax)_j \tag{1}$$

In general, there may be k indices satisfying satisfying Equation (1). Define $S = \{i_1, \ldots, i_k\}$. Consider the restriction of x on S, x^S and the restriction of A on S, A^S. Then, it is shown that satisfying the second condition of an ESS is equivalent to

$$(x^S - y)^T A^S (x^S - y) < 0, \; \forall y \in \Delta_k, \; x^S \neq y \tag{2}$$

Rather than using this characterisation, we will prove in a direct way, based on the payoff matrix of the game, that for some class of games the first condition of an ESS cannot be satisfied. This is in line with results from the literature.

4 Results

Consider the following examples:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

From left to right, the first game contains one ESS: $(0.5, 0.5)$. The second game contains two ESS's: e^1, e^2. The third game contains one ESS: e^1. Finally, the fourth game contains no ESS's. It is difficult to tell whether a certain strategy is an ESS, especially for large payoff matrices. As an extra complication, the unit simplex Δ provides uncountably many candidates.

These first results will prove to be useful.

Proposition 1.

i. If A is a positive semidefinite game and $S \subseteq \{1, \ldots, n\}$, $2 \leq |S| \leq n$, then A^S is a positive semidefinite game.

ii. If $n = 2$, A is a positive semidefinite game if and only if $a_{11} + a_{22} \geq a_{12} + a_{21}$.

Proof. Proof of i.: Let $2 \leq |S| \leq n$ and suppose A^S is not a positive semidefinite game. Then there exists $z \in \mathbb{R}_0^{|S|}, z^T A^S z < 0$. Consider the vector $\bar{z} \in \mathbb{R}_0^n$ defined as follows: $\bar{z}_i = \mathbf{1}_S \cdot z_i$, $i = 1, \ldots, n$. Clearly, $\bar{z}^T A \bar{z} < 0$, so A is not a positive semidefinite game.

Proof of ii.: ($\Rightarrow$) Take $z^T = [1, -1]$ and rewrite.
($\Leftarrow$) For every $z = (z^*, -z^*) \in \mathbb{R}_0^2$:

$$\begin{aligned} z^T Az &= {z^*}^2 a_{11} - {z^*}^2 a_{12} - {z^*}^2 a_{21} + {z^*}^2 a_{22} \\ &= {z^*}^2 (a_{11} + a_{22} - a_{12} - a_{21}) \geq 0 \end{aligned}$$ □

Proposition 2. *Let A be an $n \times n$ game and let $x \in \Delta$.*

i. $z^T Az \geq 0\ \forall z \in \mathbb{R}_0^n$ *if and only if* $(x-y)^T A(x-y) \geq 0\ \forall y$

ii. $z^T Az > 0\ \forall z \in \mathbb{R}_0^n \backslash \{0\}$ *if and only if* $(x-y)^T A(x-y) > 0\ \forall y \neq x$

Proof. We will prove the second assertion; the first one is a straightforward modification.
($\Rightarrow$) $\forall y \neq x$: $\sum_{i=1}^n (x-y)_i = 0$ and $x - y \neq 0$. So $(x-y) \in \mathbb{R}_0^n \backslash \{0\}$ thus $(x-y)^T A(x-y) > 0$.
($\Leftarrow$) Any $z \in \mathbb{R}_0^n \backslash \{0\}$ can be written as $z = \lambda(x-y)$ where $2\lambda = \sum_{i=1}^n |z_i|$, $z^+ = \lambda x, z^- = \lambda y$, for some $x, y \in \Delta$. □

Building on these results, we now claim the following.

Theorem 1 (Evolutionarily stable strategies in positive semidefinite games)**.** *If A is a positive semidefinite game then any ESS is pure.*

Proof. Let x be an ESS and let S be the support of x. Assume $2 \leq |S| \leq n$. By restriction, x^S is an ESS of A^S (for if it was not an ESS of A^S, there would be an invading strategy $y \in \Delta_{|S|}$ and we would also have an invading strategy $\bar{y} \in \Delta$ for A defined as follows: $\bar{y}_i = \mathbf{1}_S \cdot y_i$) that has full support and A^S is a positive semidefinite game, by Proposition 1, a contradiction with Proposition 2 and the fact that x interior Nash equilibrium strategy implies x is an ESS if and only if $(x-y)^T A(x-y) < 0\ \forall y \neq x$ (see e.g. Hofbauer [2011], Abakuks [1980]). It follows that $|S| = 1$. □

Theorem 1 connects the nonexistence of mixed ESS's to Länger [1993]. In this paper (Theorem 17) Länger gives a complete characterisation of 3×3 games and the existence of ESS's for a given support. Without loss of generality, it is assumed that all diagonal entries are equal to 0.

$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}$$

A necessary condition to have an ESS's with support $\{1, 2\}$ is that $a, c > 0$. From Proposition 1, this cannot hold true for positive semidefinite games. The cases with support $\{1,3\}$ and $\{2, 3\}$ are similar and the same is true for 2×2 matrices (Theorem 14). A necessary condition for a completely mixed evolutionarily stable strategy is to have $a + c > 0$, $b + d > 0$ and $c + f > 0$, which is impossible because of the same Proposition. From Hofbauer [2011], Abakuks [1980] we know that a completely mixed strategy cannot be evolutionarily stable in a positive semidefinite game.

We may wonder whether we can discard mixed strategies altogether when checking if a given pure strategy is evolutionarily stable, which is not true in general. Lemma 1 answers this question affirmatively.

Lemma 1. *If A is a positive semidefinite game then any pure ESS is strict Nash.*

Proof. Let e^i be i-th pure strategy, $1 \leq i \leq n$. We distinguish two cases:

1. $\exists j \neq i\ a_{ji} \geq a_{ii}$. We subdivide this into two other cases:

   (a) $a_{ji} > a_{ii}$ then e^i is not a Nash equilibrium strategy
   (b) $a_{ji} = a_{ii}$ then by Proposition 1 $a_{jj} \geq a_{ij}$ so e^i can be invaded by e^j.

   In either case e^i is not an ESS.

2. $\nexists j \neq i \ a_{ji} \geq a_{ii} \Leftrightarrow \forall j \neq i \ a_{ji} < a_{ii}$ then we have a strict Nash strategy and e^i is an ESS. □

The above results culminate in the following theorem.

Theorem 2. *If A is a positive semidefinite game then a strategy x is an ESS if and only if x is strict Nash.*

Proof. ($\Rightarrow$) Follows from Theorem 1 and Lemma 1.
($\Leftarrow$) Immediate. □

In general we may have 0, 1, ..., n pure ESS's, but no mixed ESS's. As an example of Theorem 2, consider Rock–Paper–Scissors:

$$z^T \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} z = 0 \text{ for all } z \in \mathbb{R}^3_0$$

From Theorem 2, there are no ESS's. Indeed, it is well known that the mixed symmetric Nash equilibrium is not evolutionarily stable.

The observation that mixed strategies can never be evolutionarily stable in positive semidefinite games is consonant with a result presented by Haigh and later by Abakuks (see Equation (2)). Given a positive semidefinite game A, we know by Proposition 1 that any restriction A^S is again positive semidefinite. And given a Nash strategy x, we have $(x^S - y)^T A^S (x^S - y) \geq 0$ for all $y \in \Delta_{|S|}$, $y \neq x^S$ from Proposition 2. Obviously this implies that Equation (2) can never be satisfied for any such x. Here we have given a rather direct proof, based on properties of the matrix, without resorting to this characterisation. A second reason to opt for the results presented here, is that we now know that we can discard all mixed strategies in positive semidefinite games, even when checking whether a given pure strategy is an ESS.

It turns out that a similar result holds true for neutrally stable strategies.

Theorem 3. *If A is a positive definite game then a strategy x is an NSS if and only if x is strict Nash.*

Proof. ($\Rightarrow$) follows directly from (slightly modified versions of) Proposition 1, Proposition 2, Theorem 1 and Lemma 1.
($\Leftarrow$) Immediate. □

We can thus conclude that if A is a positive definite game then $\Delta^{ESS} = \Delta^{NSS}$. As an example, take the 2×2 coordination game:

$$z^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z = z_1^2 + z_2^2 > 0 \text{ for all } z \in \mathbb{R}^2_0 \backslash \{0\}$$

Although the strategy (0.5, 0.5) is not neutrally stable, the game does possess two neutrally stable strategies, which are also evolutionarily stable and pure: e^1 and e^2. In fact, the coordination game is a particular case of a more general result which we obtain by combining Theorem 3 and a result following from Weibull [1995, p. 56, Proposition 2.14] for doubly symmetric games (games for which $A = A^T$).

Corollary 1 (Existence of pure ESS in doubly symmetric positive definite games)**.** *Let A be a positive definite and doubly symmetric game. Then A contains an ESS in pure strategies.*

Proof. Existence of an NSS is proved for example in Weibull [1995, p. 56, Proposition 2.14]. From Theorem 3 we know that every NSS is an ESS and that is strict Nash. □

5 Conclusion

Positive semidefinite games cannot possess mixed evolutionarily stable strategies. Finding ESS's can be done using a procedure involving only the pure strategies. In this case, being ESS is equivalent to being strict Nash. Positive definite games cannot possess mixed neutrally stable strategies. Moreover, also for positive definite games we only need pure strategies to find NSS's. In this case, being NSS is equivalent to being strict Nash and so the possible neutrally stable pure strategies are always evolutionarily stable as well. Two famous games emerge as a special case to the theorems: Rock–Paper–Scissors in the former case and the coordination game in the latter case. In doubly symmetric and positive definite games existence of an ESS is guaranteed. This is nice because evolutionarily stable strategies are asymptotically stable in the replicator dynamics rather than just Lyapunov stable.

References

Andris Abakuks. Conditions for Evolutionarily Stable Strategies. *Journal of Applied Probability*, 17(2):559–562, 1980.

D. T. Bishop and C. Cannings. Models of Animal Conflict. *Advances in Applied Probability*, 8(4):616–621, 1976.

John Haigh. Game Theory and Evolution. *Advances in Applied Probability*, 7(1):8–11, 1975.

Josef Hofbauer. Deterministic Evolutionary Game Dynamics. In Karl Sigmund, editor, *Proceedings of Symposia in Applied Mathematics*, volume 69, pages 61–79. American Mathematical Society, 2011.

Helmut Länger. A characterization of evolutionarily stable strategies. *Mathematica Pannonica*, 4(2):225–233, 1993.

Jörgen W Weibull. *Evolutionary game theory.* MIT Press, Cambridge, Ma, 1995. ISBN 0262231816.